MATHEMATICS - 1999

PART - A

Directions : *Select the most appropriate alternative A, B, C or D in questions 1–25.*

1. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to :

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$

(C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

2. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G. P. with the same common ratio, then the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) :

(A) lie on a straight line (B) lie on an ellipse

(C) lie on a circle (D) are vertices of a triangle

3. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is :

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

(C) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (D) not defined

4. The harmonic mean of the roots of the equation

$$(5 + \sqrt{2})\,x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0 \text{ is :}$$

(A) 2 (B) 4

(C) 6 (D) 8

5. The function $f(x) = \sin^4 x + \cos^4 x$ increases if :

(A) $0 < x < \frac{\pi}{8}$ (B) $\frac{\pi}{4} < x < \frac{3\pi}{8}$

(C) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (D) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

6. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents :

(A) a pair of straight lines (B) an ellipse

(C) a parabola (D) a hyperbola

7. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then :

(A) $a + b = c$ (B) $b + c = a$

(C) $a + c = b$ (D) $b = c$

8. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2 \sin x]\, dx$ is :

(A) $-\pi$
(B) 0
(C) $-\frac{\pi}{2}$
(D) $\frac{\pi}{2}$

9. Let $a_1, a_2, \ldots, a_{10}$ be in A. P. and $h_1, h_2, \ldots, h_{10}$ be in H. P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is :

(A) 2
(B) 3
(C) 5
(D) 6

10. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|, |\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$ is $30°$, then $\left|(\vec{a} \times \vec{b}) \times \vec{c}\right| =$

(A) $\frac{2}{3}$
(B) $\frac{3}{2}$
(C) 2
(D) 3

11. The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2} \text{ is :}$$

(A) zero
(B) one
(C) two
(D) infinite.

12. Let $P(a \sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to :

(A) $\frac{a^2 + b^2}{a}$
(B) $-\left(\frac{a^2 + b^2}{a}\right)$
(C) $\frac{a^2 + b^2}{b}$
(D) $-\left(\frac{a^2 + b^2}{b}\right)$

13. Let PQR be a right angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is :

(A) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
(B) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
(C) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
(D) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$

14. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ then $f(100)$ is equal to :

(A) 0 (B) 1

(C) 100 (D) −100

15. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :

(A) all integers (B) all integers except 0 and 1

(C) all integers except 0 (D) all integers except 1

16. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then :

(A) $p^2 = q^2$ (B) $p^2 = 8q^2$

(C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

17. The function $f(x) = (x^2 - 1)\left|x^2 - 3x + 2\right| + \cos(|x|)$ is NOT differentiable at :

(A) −1 (B) 0

(C) 1 (D) 2

18. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then :

(A) $a < 2$ (B) $2 < a \leq 3$

(C) $3 < a \leq 4$ (D) $a > 4$

19. A solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is :

(A) $y = 2$ (B) $y = 2x$

(C) $y = 2x - 4$ (D) $y = 2x^2 - 4$

20. $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is :

(A) 2 (B) −2

(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

21. Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ =

(A) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ (B) $\frac{1}{\sqrt{3}}(-\hat{i} - \hat{j} - \hat{k})$

(C) $\frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$ (D) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

22. If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is :

(A) 6

(B) 9

(C) 12

(D) 24

23. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to :

(A) 2

(B) -2

(C) $\frac{1}{2}$

(D) $-\frac{1}{2}$

24. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is :

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x - 9 = 0$

(D) $9x^2 - 8y^2 + 18x + 9 = 0$

25. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals :

(A) $\frac{1}{4}$

(B) $\frac{1}{7}$

(C) $\frac{1}{8}$

(D) $\frac{1}{49}$

Directions : *Question numbers 26–35 carry 3 marks each and may have more than one correct answers. All correct answers must be marked to get any credit in these questions :*

26. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1 ?

(A) $x + y = 0$

(B) $x - y = 0$

(C) $x + 7y = 0$

(D) $x - 7y = 0$

27. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\,\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then $|\vec{v}|$ is :

(A) $\left|\vec{u}\right|$

(B) $\left|\vec{u}\right| + \left|\vec{u} \cdot \vec{a}\right|$

(C) $\left|\vec{u}\right| + \left|\vec{u} \cdot \vec{b}\right|$

(D) $\left|\vec{u}\right| + \vec{u} \cdot (\vec{a} + \vec{b})$

28. For a positive integer n, let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{(2^n) - 1}$. Then :

(A) $a(100) \leq 100$ (B) $a(100) > (100)$

(C) $a(200) \leq 100$ (D) $a(200) > 100$

29. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local minimum at x =

(A) 0 (B) 1

(C) 2 (D) 3

30. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are :

(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (B) $\left(\frac{2}{5}, \frac{1}{5}\right)$

(C) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (D) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

31. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. Of these subjects, the student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two, and a 40% chance of passing in exactly two. Which of the following relations are true ?

(A) $p + m + c = \frac{19}{20}$ (B) $p + m + c = \frac{27}{20}$

(C) $pmc = \frac{1}{10}$ (D) $pmc = \frac{1}{4}$

32. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is of :

(A) order 1 (B) order 2

(C) degree 3 (D) degree 4

33. Let $S_1, S_2 \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

(A) 7 (B) 8

(C) 9 (D) 10

34. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals $\frac{9}{2}$?

(A) –4 (B) –2

(C) 2 (D) 4

35. For a positive integer n, let

$f_n(\theta) = \left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$. Then

(A) $f_2\left(\frac{\pi}{16}\right) = 1$ (B) $f_3\left(\frac{\pi}{32}\right) = 1$

(C) $f_4\left(\frac{\pi}{64}\right) = 1$ (D) $f_5\left(\frac{\pi}{128}\right) = 1$

ANSWERS

1. (C)	**2.** (A)	**3.** (B)	**4.** (B)	**5.** (B)	**6.** (C)
7. (A)	**8.** (C)	**9.** (D)	**10.** (B)	**11.** (C)	**12.** (D)
13. (B)	**14.** (A)	**15.** (B)	**16.** (D)	**17.** (D)	**18.** (A)
19. (C)	**20.** (C)	**21.** (A)	**22.** (C)	**23.** (A)	**24.** (B)
25. (A)	**26.** (A), (C)	**27.** (B), (C)	**28.** (A), (D)	**29.** (B), (D)	**30.** (B), (D)
31. (B),	**32.** (A), (C)	**33.** (B), (C), (D)	**34.** (B), (D)	**35.** (A), (B), (C), (D).	

SOLUTIONS

1. **Imp. note :** If in a complex no. $a + ib$, the ratio $a : b$ is $1 : \sqrt{3}$ then always try to convert that complex no. in ω.

Here $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Therefore, $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$

$= 4 + 5\omega^{334} + 3\omega^{365}$

$= 4 + 5\cdot(\omega^3)^{111}\cdot\omega + 3\cdot(\omega^3)^{121}\cdot\omega^2$

$= 4 + 5\omega + 3\omega^2$ $\quad\because \omega^3 = 1$

$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$

$= 1 + 2\omega + 3(1 + \omega + \omega^2) = 1 + 2\omega + 3 \times 0 \quad \because 1 + \omega + \omega^2 = 0$

$= 1 + (-1 + \sqrt{3}i) = \sqrt{3}i$ Therefore, (C) is the answer.

2. Let $\frac{x_2}{x_1} = \frac{x_3}{x_2} = r$ and $\frac{y_2}{y_1} = \frac{y_3}{y_2} = r$

$\Rightarrow x_2 = x_1 r,\ x_3 = x_1 r^2$ and $y_2 = y_1 r$ and $y_3 = yr^2$.

again, $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix}$

Using $R_3 \to R_3 - rR_2$ and $R_2 \to R_2 - rR_1$ ∵ 3, one's ⇒ 2, zero's ⇒ 20

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1-r \\ 0 & 0 & 1-r \end{vmatrix} = 0 \quad \because \; R_2 \text{ and } R_3 \text{ are identicale}$$

Thus $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on a straight line.
Therefore, (A) is answer.

3. Let $y = 2^{x(x-1)}$ where $y \geq 1$ as $x \geq 1$.

taking $\log_2$ of both side

$$\log_2 y = \log_2 2^{x(x-1)}$$

$$\Rightarrow \quad \log_2 y = x(x-1) \qquad \because \quad \log_a a^x = x$$

$$\Rightarrow \quad x^2 - x - \log_2 y = 0$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1 + 4\log_2 y}}{2}$$

For $y \geq 1, \log_2 y \geq 0 \Rightarrow 4\log_2 y \geq 0 \Rightarrow 1 + 4\log_2 y \geq 1$

$$\Rightarrow \quad \sqrt{1 + 4\log_2 y} \geq 1$$

$$\Rightarrow \quad -\sqrt{1 + 4\log_2 y} \leq -1$$

$$\Rightarrow \quad 1 - \sqrt{1 + 4\log_2 y} \leq 0$$

but $\quad x \geq 1$

so $\quad x = 1 - \sqrt{1 + 4\log_2 y}$ is not possible.

therefore $x = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 y})$

$$\Rightarrow \quad f^{-1}(y) = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 y}) \qquad \because x = f^{-1}(y)$$

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$$

Therefore, (B) is the Answer.

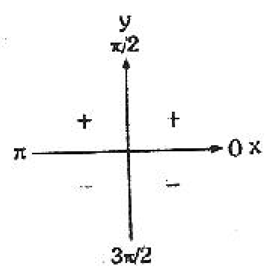
4. Let α, β be the root of given quardratic equation. Then

$$S = \alpha + \beta = \frac{4 + \sqrt{5}}{5 + \sqrt{2}} \qquad \text{and} \qquad \alpha\beta = \frac{8 + 2\sqrt{5}}{5 + \sqrt{2}}$$

Again, H be the Harmonic mean between α and β, then

$$H = \frac{2\alpha\beta}{\alpha + \beta} = \frac{16 + 4\sqrt{5}}{4 + \sqrt{5}} = 4$$ Therefore, (B) is the answer.

5. $f(x) = \sin^4 x + \cos^4 x$



Differentiating w.r.t. x, we get

$$f'(x) = 4\sin^3 x \cdot \cos x - 4\cos^3 x \cdot \sin x$$

$$= 4\sin x \cos x(\sin^2 x - \cos^2 x)$$

$$= 2 \cdot \sin 2x(-\cos 2x)$$

$$= -\sin 4x$$

Now, $f'(x) > 0$ if $\sin 4x < 0$

$\Rightarrow \quad \pi < 4x < 2\pi$

$\Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$ ...(1) and (A) is wrong. $\therefore \quad 0 < x < 3\pi/8$

$\Rightarrow \quad$ (A) is not proper subset of (1)

Again (B) is the answer since (B) is proper subset of (1)

Again (C), $\frac{3\pi}{8} < x < \frac{5\pi}{8}$, is not the answer because C is not proper subset of (1)

Again (D) is not answer.

6. $x = t^2 + t + 1$ ...(1)

$y = t^2 - t + 1$ ...(2)

Imp. note : In this, direct substitution in terms of y or x of t is a typical method. So we will use here slight different way.

subtract (2) from (1)

$$x - y = 2t$$

Thus, $$x = t^2 + t + 1$$

$$\Rightarrow \quad x = \left(\frac{x-y}{2}\right)^2 + \left(\frac{x-y}{2}\right) + 1$$

$$\Rightarrow \quad 4x = (x-y)^2 + 2x - 2y + 4$$

$$\Rightarrow \quad (x-y)^2 = 2(x + y - 2)$$

$$\Rightarrow \quad x^2 + y^2 - 2xy = 2x + 2y - 4$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$$

Now $\Delta = 11.4 + 2.(-1)(-1)(-1) - 1 \times (-1)^2 - 1 \times (-1)^2 - 4(-1)^2$

$= 4 - 2 - 1 - 1 - 4$

$= -4$ therefore $\Delta \neq 0$

and $ab - h^2 = 11 - (1)^2 = 1 - 1$

$= 0$ so it is equation of a parabola, therefore (C) is the answer.

7. It is given that $\tan P/2$ and $\tan Q/2$ are the roots of the quadratic equation $ax^2 + bx + c = 0$ and $\angle R = \pi/2$

so $$\tan P/2 + \tan Q/2 = -b/a$$

$$\tan P/2 \tan Q/2 = c/a$$

Now $$P + Q + R = 180°$$

$$\Rightarrow \quad P + Q = 90°$$

$$\Rightarrow \quad \frac{P+Q}{2} = 45°$$

taking tan of both sides

$$\tan\left(\frac{P+Q}{2}\right) = \tan 45°$$

$$\Rightarrow \quad \frac{\tan P/2 + \tan Q/2}{1 - \tan P/2 \cdot \tan Q/2} = 1$$

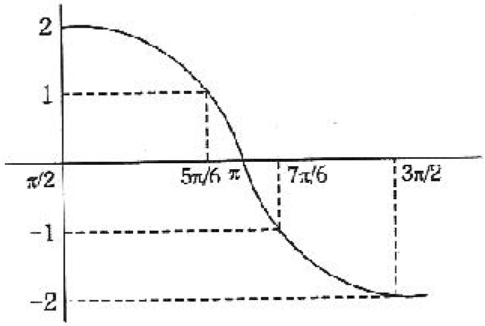
$$\Rightarrow \quad \frac{-b/a}{1-c/a} = 1$$

$$\Rightarrow \quad \frac{-b/a}{\frac{a-c}{a}} = 1 \Rightarrow \quad \frac{-b}{a-c} = 1$$

$$\Rightarrow \quad -b = a - c$$

$\Rightarrow$ $a - b = c$. Therefore, (A) is the answer.

8. The graph of $y = 2\sin x$ for $\pi/2 \le x \le 3\pi/2$ is given in Fig. From the graph it is clear that



$$[2\sin x] = \begin{cases} 2 & \text{if} \quad x = \pi/2 \\ 1 & \text{if} \quad \pi/2 < x \le 5\pi/6 \\ 0 & \text{if} \quad 5\pi/6 < x \le \pi \\ -1 & \text{if} \quad \pi < x \le 7\pi/6 \\ -2 & \text{if} \quad 7\pi/6 < x \le 3\pi/2 \end{cases}$$

Therefore,

$$\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx$$

$$= \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0\, dx + \int_{\pi}^{7\pi/6} (-1)\, dx + \int_{7\pi/6}^{3\pi/2} (-2)\, dx$$

$$= [x]_{\pi/2}^{5\pi/6} + [-x]_{\pi}^{7\pi/6} + [-2x]_{7\pi/6}^{3\pi/2}$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + \left(\frac{-7\pi}{6} + \pi\right) + \left(\frac{-2\cdot 3\pi}{2} + \frac{2\cdot 7\pi}{6}\right)$$

$$= \pi\left(\frac{5}{6} - \frac{1}{2}\right) + \pi\left(1 - \frac{7}{6}\right) + \pi\left(\frac{7}{3} - 3\right)$$

$$= \pi\left(\frac{5-3}{6}\right) + \pi\left(-\frac{1}{6}\right) + \pi\left(\frac{7-9}{3}\right) = \frac{-\pi}{2}$$

Therefore, (C) is the answer.

9. $a_1, a_2, a_3 \ldots a_{10}$ be in A.P.

so, $\quad a_{10} = a_1 + 9d$

$\Rightarrow \quad 3 = a_1 + 9d$

$\Rightarrow \quad 3 = 2 + 9d$

$\Rightarrow \quad d = 1/9$

Now, $\quad a_4 = a_1 + 3d$

$\Rightarrow \quad a_4 = 2 + 3(1/9) = 2 + 1/3 - 7/3$

Again $h_1, h_2, h_3 \ldots h_{10}$ be in H.P.

$\Rightarrow \quad \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \ldots \frac{1}{h_{10}}$ be in A.P.

$h_1 = 2, \quad h_{10} = 3$ (given).

so, $$\frac{1}{h_{10}} = \frac{1}{h_1} + 9d_1$$

$$\Rightarrow \quad \frac{1}{3} = \frac{1}{2} + 9d_1$$

$$\Rightarrow \quad \frac{1}{3} - \frac{1}{2} = 9d_1$$

$$\Rightarrow \quad -\frac{1}{6} = 9d_1$$

$$\Rightarrow \quad d_1 = -\frac{1}{54}$$

Now, $\frac{1}{h_7} = \frac{1}{h_1} + 6d_1$

$$\frac{1}{h_7} = \frac{1}{2} + \frac{6 \times 1}{-54}$$

$$\frac{1}{h_7} = \frac{1}{2} - \frac{1}{9}$$

$$\frac{1}{h_7} = \frac{9-2}{18}$$

$$h_7 = \frac{18}{7}$$

So $a_4\, h_7 = \frac{7}{3} \times \frac{18}{7} = 6$. Therefore, (D) is the answer.

10. **Imp. note :** In this Question vector $\vec{c}$ is not given, therefore, we cannot apply the formulae of $\vec{a} \times \vec{b} \times \vec{c}$ (vector triple product).

Now $\quad \left|(\vec{a} \times \vec{b}) \times \vec{c}\right| = \left|\vec{a} \times \vec{b}\right| \left|\vec{c}\right| \sin 30^\circ$

So we need now $\left|\vec{a} \times \vec{b}\right|$ and $\left|\vec{c}\right|$

again $$\left|\vec{a} \times \vec{b}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$$\Rightarrow \quad \left|\vec{a} \times \vec{b}\right| = \sqrt{2^2 + (-2)^2 + 1} = \sqrt{4+4+1} = \sqrt{9} = 3$$

Next, $\left|\vec{c} - \vec{a}\right| = 2\sqrt{2}$

$\Rightarrow \quad \left|\vec{c} - \vec{a}\right|^2 = 8$

$\Rightarrow \quad (\vec{c} - \vec{a}) \cdot (\vec{c} - \vec{a}) = 8$

$\Rightarrow \quad \vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{a} - \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{a} = 8$

$\Rightarrow \quad \left|\vec{c}\right|^2 + \left|\vec{a}\right|^2 - 2\,\vec{a} \cdot \vec{c} = 8$

$\Rightarrow \quad \left|\vec{c}\right|^2 + 9 - 2\left|\vec{c}\right| = 8 \quad \therefore \vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ (given), $\vec{a} \cdot \vec{c} = \left|\vec{c}\right|$ (given)

$\Rightarrow \quad \left|\vec{c}\right|^2 - 2\left|\vec{c}\right| + 1 = 0$

$\Rightarrow \quad \left(\left|\vec{c}\right| - 1\right)^2 = 0$

$\Rightarrow \quad \left|\vec{c}\right| = 1$

Now putting in

$$\left|(\vec{a} \times \vec{b}) \times \vec{c}\right| = \left|\vec{a} \times \vec{b}\right|\left|\vec{c}\right| \sin 30^\circ$$

$= (3)(1) \cdot \left(\frac{1}{2}\right) = \frac{3}{2}$. Therefore, (B) is the Ans.

11. From function it is clear that

(1) $x(x+1) > 0 \quad \because$ Domain of square root function.

(2) $x^2 + x + 1 \ge 0 \quad \because$ Domain of square root function.

(3) $x^2 + x + 1 \le 1 \quad \because \quad \sqrt{x^2 + x + 1} \le 1 \quad \therefore$ Domain of $\sin^{-1}$ function.

From (2) and (3)

$0 \le x^2 + x + 1 \le 1 \cap x^2 + x \ge 0$

$\Rightarrow \quad 0 \le x^2 + x + 1 \le 1 \cap x^2 + x + 1 \ge 1$

$\Rightarrow \quad x^2 + x + 1 = 1$

$\Rightarrow \quad x^2 + x = 0$

$\Rightarrow \quad x(x + 1) = 0$

$\Rightarrow \quad x = 0,\ x = -1$, Therefore, (C) is the answer.

12. Firstly we obtain the slope of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$.

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{b^2}{a^2}\cdot\frac{x}{y}$$

Slope of the normal at the point $(a \sec\theta, b \tan\theta)$ will be equal to

$$-\left(\frac{dx}{dy}\right)_{(a\sec\theta,\, b\tan\theta)} = -\frac{a^2}{b^2}\,\frac{b\tan\theta}{a\sec\theta} = -\frac{a}{b}\sin\theta$$

$\therefore$ equation of normal at $(a \sec\theta, b \tan\theta)$ is

$$y - b\tan\theta = -\frac{a}{b}\sin\theta\,(x - a\sec\theta)$$

$$\Rightarrow \qquad (a\sin\theta)\,x + by = (a^2 + b^2)\tan\theta$$

$$\Rightarrow \qquad ax + b\,\mathrm{cosec}\, y = (a^2 + b^2)\sec 0 \qquad \ldots(1)$$

Similarly equation of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\phi, b \tan\phi)$ is

$$ax + b\,\mathrm{cosec}\,\phi\, y = (a^2 + b^2)\sec\phi \qquad \ldots(2)$$

Subtracting (2) from (i) we get

$b\,(\mathrm{cosec}\,\theta - \mathrm{cosec}\,\phi)\, y = (a^2 + b^2)(\sec\theta - \sec\phi)$

$$\Rightarrow \qquad y = \frac{a^2 + b^2}{b}\cdot\frac{\sec 0 - \sec\phi}{\mathrm{cosec}\,\theta - \mathrm{cosec}\,\phi}$$

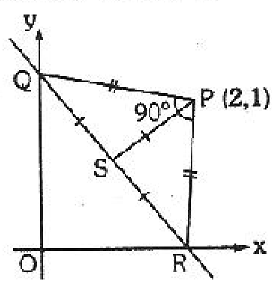
But $\frac{\sec\theta - \sec\phi}{\mathrm{cosec}\,\theta - \mathrm{cosec}\,\phi} = \frac{\sec\theta - \sec(\pi/2 - \theta)}{\mathrm{cosec}\,\theta - \mathrm{cosec}\,(\pi/2 - \theta)}$ $\qquad [\phi + \theta = \pi/2]$

$$= \frac{\sec\theta - \mathrm{cosec}\,\theta}{\mathrm{cosec}\,\theta - \sec\theta} = 1$$

Thus, $y = -\frac{a^2 + b^2}{b}$ i.e. $k = -\frac{a^2 + b^2}{b}$ therefore (D) is the ans.

13. Let S be the mid-point of QR and $\Delta\, PQR$ is isoseles (given).

Therefore $PS \perp QR$ and S is mid-point of hypotenuse, therefore, S is equidistant from P, Q, R $\quad \therefore\ PS = QS = RS$



Now $\quad \angle P = 90°$ and $\angle Q = \angle R$

But $\quad \angle P + \angle Q + \angle R = 180°$

So, $\quad 90° + \angle Q + \angle R = 180°$

$\therefore \quad \angle Q = 45°$ and $\angle R = 45°$

Now slope of QR is -2 (given).

$QR \perp PS \quad \therefore \quad$ slope of PS is $+1/2$.

Now Let m be the slope of PQ.

Therefore, $\quad \tan(\pm 45°) = \frac{m - 1/2}{1 - m(-1/2)}$

$$\Rightarrow \qquad \pm 1 = \frac{2m - 1}{2 + m}$$

$$\Rightarrow \qquad m = 3, -1/3$$

∴ Equation of PQ and PR are

$$y - 1 = 3(x - 2) \text{ and } y - 1 = -\frac{1}{3}(x - 2) \text{ or } 3(y - 1) + (x - 2) = 0.$$

Therefore, joint equations of PQ and PR are

$$[3(x - 2) - (y - 1)][(x - 2) + 3(y - 1)] = 0$$

$$\Rightarrow \quad 3(x - 2)^2 - 3(y - 1)^2 + 8(x - 2)(y - 1) = 0$$

$$\Rightarrow \quad 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$$

Therefore, Ans. is (B).

14. $f(x) = \begin{vmatrix} 1 & x & x + 1 \\ 2x & x(x - 1) & (x + 1)x \\ 3x(x - 1) & x(x - 1)(x - 2) & (x + 1)x(x - 1) \end{vmatrix}$

Imp. Note : Observe in R_1 that $a_{11} + a_{12} = a_{13}$ Check this trend in R_2 and R_3

apply $R_3 \to R_3 - (R_1 + R_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x - 1) & 0 \\ 3x(x - 1) & x(x - 1)(x - 2) & 0 \end{vmatrix} = 0$$

$\therefore f(x) = 0 \Rightarrow f(100) = 0$. Therefore, (A) is the answer.

15. **Imp. note :** All Integers are critical point for greatest Integer function.

So, Case 1 : $x \in I$,

$$f(x) = [x]^2 - [x^2] = x^2 - x^2 = 0$$

Case 2 : $x \notin I$.

if $0 < x < 1$, then $[x] = 0$

and $0 < x^2 < 1$, then $[x^2] = 0$

Next, if $1 < x^2 < 2 \Rightarrow 1 < x < \sqrt{2} \Rightarrow [x] = 1$ and $[x^2] = 1$

Therefore, $f(x) = [x]^2 - [x^2] = 0$ if $1 < x < \sqrt{2}$

Therefore, $f(x) = 0$, if $0 \le x < \sqrt{2}$

This shows that $f(x)$ is continuous at $x = 1$

Therefore, $f(x)$ is discontinuous in $(-\infty, 0) \cup [\sqrt{2}, \infty)$ on many other points.

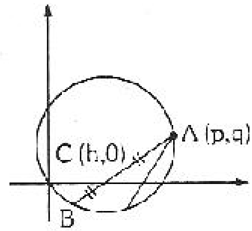
Therefore, (B) is the answer.

16. **Imp. note :** In solving a line and a circle there often generate a quadratic equation and further we have to apply condition of Discriminant so question convert from coordinate to quadratic equation.

Ans. From equation of circle it is clear that circle passes through origin.

Let AB is chord of the circle

$A \equiv (p, q)$. C is mid-point (h, o)



Then coordinates of B are $(-p + 2h, -q)$.
And B lies on the circle

$$x^2 + y^2 = px + qy, \text{ we have}$$

$$(-p + 2h)^2 + (-q)^2 = p(-p + 2h) + q(-q)$$

$$\Rightarrow \quad p^2 + 4h^2 - 4ph + q^2 = -p^2 + 2ph - q^2$$

$$\Rightarrow \quad 2p^2 + 2q^2 - 6ph + 4h^2 = 0$$

$$\Rightarrow \quad 2h^2 - 3ph + p^2 + q^2 = 0 \qquad \ldots(1)$$

There are given two distinct chords which are bisected at x axis then there will be two distinct values of h satisfying (1).
So discriminant of this quadratic equation must be > 0.

$$\Rightarrow \quad D > 0$$

$$\Rightarrow \quad (-3p)^2 - 4 \cdot 2 (p^2 + q^2) > 0$$

$$\Rightarrow \quad 9p^2 - 8p^2 - 8q^2 > 0$$

$$\Rightarrow \quad p^2 - 8q^2 > 0$$

$$\Rightarrow \quad p^2 > 8q^2$$ Therefore, (D) is the Answer.

17. Function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|) \qquad \ldots(1)$

> **Imp. note :** In differentiability of $|f(x)|$ we have to consider critical points for which $f(x) = 0$.

$|x|$ is not differentiable at $x = 0$

but
$$\cos|x| = \begin{cases} \cos(-x) & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$$

$$\Rightarrow \quad \cos|x| = \cos x \quad \text{if } x < 0$$
$$= \cos x \quad \text{if } x \geq 0.$$ Therefore it is differentiable at $x = 0$.

Next, $|x^2 - 3x + 2| = |(x - 1)(x - 2)|$

$$= \begin{cases} (x-1)(x-2) & \text{if } x < 1 \\ (x-1)(2-x) & \text{if } 1 \leq x < 2 \\ (x-1)(x-2) & \text{if } 2 \leq x \end{cases}$$

Therefore,

$$f(x) = \begin{cases} (x^2 - 1)(x-1)(x-2) + \cos x & \text{if } -\infty < x < 1 \\ -(x^2 - 1)(x-1)(x-2) + \cos x & \text{if } 1 \leq x < 2 \\ (x^2 - 1)(x-1)(x-2) + \cos x & \text{if } 2 \leq x < \infty \end{cases}$$

Now $x = 1, 2$ are critical point for differentiability.
Because $f(x)$ is differentiable on other points in its domain.

Differentiability at x = 1

$$L\, f'(1) = \lim_{x \to 1-0} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1-0} \left[(x^2 - 1)(x-2) + \frac{\cos x - \cos 1}{x-1}\right]$$

$$= 0 - \sin 1 = -\sin 1$$

$$\because \quad \lim_{x \to 1-0} \frac{\cos x - \cos 1}{x-1} = \frac{d}{dx}(\cos x)\ at\ x = 1-0$$

$$= -\sin x \quad at\ x = 1-0$$

$$= -\sin x \quad at\ x = 1$$

$$= -\sin 1$$

and $$Rf'(1) = \lim_{x \to 1+0} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1+} \left[-(x^2 - 1)(x-2) + \frac{\cos x - \cos 1}{x-1}\right]$$

$$= 0 - \sin 1 = -\sin 1 \qquad \text{(same approach)}$$

$\because$ $L\, f'(1) = R\, f'(1)$. Therefore, function is differentiable at $x = 1$.

Again $$Lf'(2) = \lim_{x \to 2-0} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2-0} \left[-(x^2 - 1)(x-1) + \frac{\cos x - \cos 2}{x-2}\right]$$

$$= -(4-1)(2-1) - \sin 2 = -3 - \sin 2$$

and $$R\, f'(2) = \lim_{x \to 2+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2+} \left[(x^2 - 1)(x-1) + \frac{\cos x - \cos 2}{x-2}\right]$$

$$= (2^2 - 1)(2-1) - \sin 2 = 3 - \sin 2$$

So $L\, f'(2) \neq R\, f'(2)$, f is not differentiable at $x = 2$.

Therefore, (D) is the Ans

18. Both root less than 3 (given).

$\Rightarrow \quad \alpha < 3,\ \beta < 3$ ...(A)

$\Rightarrow \quad S = \alpha + \beta < 6$

$\Rightarrow \quad \dfrac{\alpha + \beta}{2} < 3$

$\Rightarrow \quad +\dfrac{2a}{2} < 3 \quad \Rightarrow \quad a < 3$ ...(1)

again $P = \alpha\beta$

$\Rightarrow \quad P < 9$

$\Rightarrow \quad \alpha\beta < 9$

$\Rightarrow \quad a^2 + a - 3 < 9$

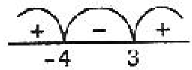
$\Rightarrow \quad a^2 + a - 12 < 0$

$\Rightarrow \quad a^2 + 4a - 3a - 12 < 0$

$\Rightarrow \quad a(a+4) - 3(a+4) < 0$

$\Rightarrow \quad (a-3)(a+4) < 0$

$\Rightarrow$



$\Rightarrow \quad -4 < a < 3$ ...(2)

again $D = B^2 - 4AC \geq 0$

$\Rightarrow \quad (-2a)^2 - 4 \cdot 1(a^2 + a - 3) \geq 0$

$\Rightarrow \quad 4a^2 - 4a^2 - 4a + 12 \geq 0$

$\Rightarrow \quad -4a + 12 \geq 0$

$\Rightarrow \quad a - 3 \leq 0$

$\Rightarrow \quad a \leq 3$ ...(3)

again, $a\, f(3) > 0$

$\Rightarrow \quad 1 \cdot f(3) > 0$

$\Rightarrow \quad (3)^2 - 2a(3) + a^2 - a - 3 > 0$

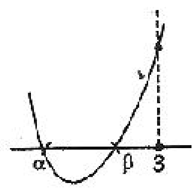
$\Rightarrow \quad 9 - 6a + a^2 + a - 3 > 0$

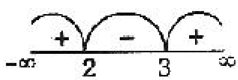
$\Rightarrow \quad a^2 - 5a + 6 > 0$

$\Rightarrow \quad a^2 - 3a - 2a + 6 > 0$

$\Rightarrow \quad a(a-3) - 2(a-3) > 0$

$\Rightarrow \quad (a-2)(a-3) > 0$





$\therefore \quad a \in (-\infty, 2) \cup (3, \infty)$ ...(4)

Collecting (1), (2), (3) and (4)

$\Rightarrow \quad a \in (-4, 2)$. Therefore, (A) is the answer.

Imp. Note : There is correction in Ans. $a < 2$ should be $-4 < a < 2$

19. Given differential equation is

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0 \quad ...(1)$$

(A) $\quad y = 2 \Rightarrow \quad \frac{dy}{dx} = 0$

putting in (1)

$$(0)^2 - x \cdot (0) + y = 0$$

$\Rightarrow \quad y = 0$ which is not satisfied.

(B) $y = 2x \Rightarrow \frac{dy}{dx} = 2$

putting in (1)

$$(2)^2 - x \cdot 2 + y = 0$$

$\Rightarrow$ $4 - 2x + y = 0$

$\Rightarrow$ $y = 2x - 4$ which is not satisfied but (C) is itself the answer.

(D) $y = 2x^2 - 4$

$$\frac{dy}{dx} = 4x$$

putting in (1)

$$\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) - y = 0$$

$\Rightarrow$ $(4x)^2 - x \cdot 4x + y = 0$

$\Rightarrow$ $y = 0$ which is not satisfied.

Therefore, (C) is the answer.

20. $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$

> **Imp. Note :** In trigonometry try to make all trigonometric functions in same angle. It is called **3rd Golden Rule** of trigonometry.

$$= \lim_{x \to 0} \frac{x \frac{2 \tan x}{1 - \tan^2 x} - 2x . \tan x}{(2 \sin^2 x)^2}$$

$$= \lim_{x \to 0} \frac{2x \tan x \left[\frac{1}{1 - \tan^2 x} - 1\right]}{4 . \sin^4 x}$$

$$= \lim_{x \to 0} \frac{2x \tan x . \left[\frac{1 - 1 + \tan^2 x}{1 - \tan^2 x}\right]}{4 \sin^4 x}$$

$$= \lim_{x \to 0} \frac{1 . x . \tan^3 x}{2 . \sin^4 x (1 - \tan^2 x)} = \lim_{x \to 0} \frac{1}{2} \frac{x . \left(\frac{\tan x}{x}\right)^3 . x^3}{\sin^4 x (1 - \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{1 \left(\frac{\tan x}{x}\right)^3}{2\left(\frac{\sin x}{x}\right)^4 . (1 - \tan^2 x)} = \frac{1 . (1)^3}{2 . (1)^4 . (1 - 0)} = \frac{1}{2}$$

Therefore, (C) is the answer.

21. It is given that $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, we take

$$\vec{c} = p\vec{a} + q\vec{b} \qquad \ldots(1)$$

where p, q are scalars.

again $\vec{c} \perp \vec{a}$, (given)

$$\Rightarrow \quad \vec{c} \cdot \vec{a} = 0$$

taking dot product of $\vec{a}$ in (1)

$$\Rightarrow \quad \vec{c} \cdot \vec{a} = p\,\vec{a} \cdot \vec{a} + q\,\vec{b} \cdot \vec{a} \qquad \because \vec{a} = 2\hat{i} + \hat{j} + \hat{k},$$

$$\Rightarrow \quad 0 = p\left|\vec{a}\right|^2 + q\left|\vec{b} \cdot \vec{a}\right| \qquad \Rightarrow \left|\vec{a}\right| = \sqrt{2^2 + 1 + 1} = \sqrt{6}$$

$$\Rightarrow \quad 0 = p \cdot 6 + q\,.3 \qquad \vec{a} \cdot \vec{b} = (2\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k})$$

$$\Rightarrow \quad q = -2p \qquad = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot (-1) = 4 - 3 = 3$$

putting in (1)

$$\Rightarrow \quad \vec{c} = p\vec{a} + \vec{b}(-2p)$$

$$\Rightarrow \quad \vec{c} = p\vec{a} - 2p\vec{b}$$

$$\Rightarrow \quad \vec{c} = p\left(\vec{a} - 2\vec{b}\right)$$

$$\Rightarrow \quad \vec{c} = p\,\{(2\hat{i} + \hat{j} + \hat{k}) - 2(\hat{i} + 2\hat{j} - \hat{k})\}$$

$$\Rightarrow \quad \vec{c} = p\,\{-3\hat{j} + 3\hat{k}\}$$

again $\left|\vec{c}\right| = 1$ (given) $\Rightarrow \left|\vec{c}\right| = p\sqrt{(-3)^2 + 3^2}$

$$\Rightarrow \quad \left|\vec{c}\right|^2 = p^2(\sqrt{18})^2$$

$$\Rightarrow \quad \left|\vec{c}\right|^2 = p^2 \cdot 18$$

$$\Rightarrow \quad 1 = p^2 \cdot 18 \Rightarrow p^2 = \frac{1}{18} \Rightarrow p = \pm\frac{1}{3\sqrt{2}}$$

$$\therefore \quad \vec{c} = \pm\frac{1}{\sqrt{2}}(-\hat{j} - \hat{k})$$

Therefore, (A) is the answer.

22. We have $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{m(m-1)}{2} \cdot x^2 + \ldots\right]$

$$\left[1 - nx + \frac{n(n-1)}{2}x^2 - \ldots\right]$$

$$= 1 + (m - n)x + \left[\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn\right]x^2 + \ldots$$

term containing power of $x \geq 3$.

(coefficient of $x = 3$ given) ...(1)

Now, $m - n = 3$

and $\frac{1}{2}m(m-1) + \frac{1}{2}n(n-1) - mn = -6$

or $m(m-1) + n(n-1) - 2mn = -12$

$\Rightarrow$ $m^2 - m + n^2 - n - 2mn = -12$

$\Rightarrow$ $(m-n)^2 - (m+n) = -12$

$\Rightarrow$ $m + n = 9 + 12 = 21$ ...(2)

Solving (1) and (2)

$m = 12$. Therefore, (C) is the answer.

23. $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ ...(1)

$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos(\pi - x)}$ $\because \int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 - \cos x}$ ...(2)

adding (1) and (2)

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x}\right) dx$$

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{2}{1 - \cos^2 x}\right) dx$$

$$2I = 2\int_{\pi/4}^{3\pi/4} \frac{1}{\sin^2 x}\,dx$$

$$I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\,dx = [-\cot x]_{\pi/4}^{3\pi/4} = \left[-\cot\frac{3\pi}{4} + \cot\frac{\pi}{4}\right]$$

$I = -(-1) + 1 = 2$. Therefore (A) is the answer

24. Let h, k be point whose chord of contact w.r.t. to hyperbola $x^2 - y^2 = 9$ is $x = 9$.

We know that chord of contact of (h, k) w.r.t. hyperbola $x^2 - y^2 = 9$ is

$T = 0 \Rightarrow$ $h \cdot x + k(-y) - 9 = 0$

$\therefore$ $hx - ky - 9 = 0$ but it is the equation of the line $x = 9$.

This is possible when $h = 1, k = 0$ (by comparing both equations). Again equation of pair of tangent is $T^2 = SS_1$

$\Rightarrow$ $(x-9)^2 = (x^2 - y^2 - 9)(1^2 - 0^2 - 9)$

$\Rightarrow$ $x^2 - 18x + 81 = (x^2 - y^2 - 9)(-8)$

$\Rightarrow$ $x^2 - 18x + 81 = -8x^2 - 8y^2 + 72$

$\Rightarrow$ $9x^2 - 8y^2 - 18x + 9 = 0$.

Therefore, (B) is the answer.

25. **Imp. point :** power of prime numbers have cyclic numbers in their unit place.

$7^1 = 7, 7^2 = 49, 7^3 = 343, \ 7^4 = 2401$ ....

Therefore, for 7^r, $r \in N$ the no. ends at unit place 7, 9, 3, 1, 7, ...

$\therefore$ $7^m + 7^n$ will be divisible by 5 if it end at 5 or 0.

but it cannot end at 5

and also cannot end at 0

For this m and n should be as follows :

	m	n
1	$4r$	$4r+2$
2	$4r+1$	$4r+3$
3	$4r+2$	$4r$
4	$4r+3$	$4r+1$

For any given value of m, there will be 25 values of n.

Hence, the probability of the required event is $\dfrac{100 \times 25}{100 \times 100} = \dfrac{1}{4}$

Therefore (A) is the ans.

26. Let equation of line L_1 be $y = mx$. Intercepts made by L_1 and L_2 on the circle will be equal i.e. L_1 and L_2 are at the same distance from the centre of the circle.

Centre of the given circle is $(1/2, -3/2)$. Therefore,

$$\frac{|1/2 - 3/2 - 1|}{\sqrt{1-1}} = \left|\frac{3m/2 + 1/2}{\sqrt{m^2+1}}\right| \quad \Rightarrow \quad \frac{2}{\sqrt{2}} = \frac{|3m+1|}{2\sqrt{m^2+1}}$$

$$\Rightarrow \quad 8(m^2+1) = (3m+1)^2 \qquad \Rightarrow \quad m^2 + 6m - 7 = 0$$

$$\Rightarrow \quad (m+7)(m-1) = 0 \qquad \Rightarrow \quad m = -7, \ m = 1$$

Thus two chords are $y + 7x = 0$ and $y - x = 0$

Therefore, (B) and (C) is the Ans.

27. Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $\vec{a}$ and $\vec{b}$ are non-collinear, $\theta \neq 0$ and $\theta \neq \pi$.

We have $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$$= \cos\theta \quad |\vec{a}| = 1, \quad |\vec{b}| = 1 \text{ given}$$

Now, $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$

Taking modulus

$$|\vec{u}| = \left|\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}\right|$$

$$\Rightarrow \quad \left|\vec{u}\right|^2 = \left|\vec{a} - \left(\vec{a}\cdot\vec{b}\right)\vec{b}\right|^2$$

$$\Rightarrow \quad \left|\vec{u}\right|^2 = \left|\vec{a} - \cos\theta\,\vec{b}\right|^2$$

$$\Rightarrow \quad \left|\vec{u}\right|^2 = \left|\vec{a}\right|^2 + \cos^2\theta\left|\vec{b}\right|^2 - 2\cos\theta\,(\vec{a}\cdot\vec{b})$$

$$\Rightarrow \quad \left|\vec{a}\right|^2 = 1 + \cos^2\theta - 2\cos^2\theta$$

$$\Rightarrow \quad \left|\vec{u}\right|^2 = 1 - \cos^2\theta$$

$$\Rightarrow \quad \left|\vec{u}\right|^2 = \sin^2\theta$$

Also $\vec{v} = \vec{a}\times\vec{b}$ (given)

$$\Rightarrow \quad \left|\vec{v}\right| = \left|\vec{a}\times\vec{b}\right|$$

$$\Rightarrow \quad \left|\vec{v}\right|^2 = \left|\vec{a}\times\vec{b}\right|^2$$

$$\Rightarrow \quad \left|\vec{v}\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2\sin^2\theta$$

$$\Rightarrow \quad \left|\vec{v}\right|^2 = \sin^2\theta$$

$$\therefore \quad \left|\vec{u}\right|^2 = \left|\vec{v}\right|^2$$

Also, $\vec{u}\cdot\vec{a} = \left[\vec{a} - \left(\vec{a}\cdot\vec{b}\right)\vec{b}\right]\cdot\vec{a}$

$$= \vec{a}\cdot\vec{a} - \left(\vec{a}\cdot\vec{b}\right)\left(\vec{b}\cdot\vec{a}\right)$$

$$= \left(\vec{a}\right)^2 - \cos^2\theta$$

$$= 1 - \cos^2\theta = \sin^2\theta$$

$$\therefore \left|\vec{u}\right| + \left|\vec{u}\cdot\vec{a}\right| = \sin\theta + \sin^2\theta \neq \left|\vec{u}\right|$$

Next $\vec{u}\cdot\vec{b} = \left(\vec{a} - \left(\vec{a}\cdot\vec{b}\right)\cdot\vec{b}\right)\cdot\vec{b}$

$$= \vec{u}\cdot\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\left(\vec{b}\cdot\vec{b}\right)$$

$$= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b}\left|\vec{b}\right|^2$$

$$= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 0 \qquad \ldots(1)$$

$\therefore \qquad \left|\vec{u}\right| - \left|\vec{u}\cdot\vec{b}\right| = \left|\vec{u}\right| + 0 = \left|\vec{u}\right| = \left|\vec{u}\right|$

Also $\qquad \vec{u}\cdot\left(\vec{a}+\vec{b}\right) = \vec{u}\cdot\vec{a} + \vec{u}\cdot\vec{b} = \vec{u}\cdot\vec{a}$

$\Rightarrow \qquad \left|\vec{u}\right| + \vec{u}\cdot\left(\vec{a}+\vec{b}\right) = \left|\vec{u}\right| + \vec{u}\cdot\vec{a} \neq \left|\vec{v}\right|$

Therefore, (A) and (D) are not the Ans. and (B) and (C) are the Ans.

28. $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n - 1}$ (given)

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \ldots \frac{1}{7}\right) + \left(\frac{1}{8} + \ldots \frac{1}{15}\right) + \ldots \left(\frac{1}{2^{n-1}+1} \ldots \frac{1}{2^n - 1}\right)$$

$$< 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \ldots \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \ldots \frac{1}{8}\right)$$

$$+ \ldots \left(\frac{1}{2^{n-1}-1} + \frac{1}{2^{n-1}+1} + \ldots \frac{1}{2^{n-1}+1}\right)$$

$$= 1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \ldots \frac{2^{n-1}}{2^{n-1}} = \frac{1+1+1+1+\ldots+1}{(n)\text{ times}} = n$$

Thus, $a(100) < 100$. Therefore, (A) is the Ans.

Next,

$$a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots \frac{1}{8}\right) + \ldots \frac{1}{2^{n-1}+1} + \ldots \frac{1}{2^n - 1}$$

$$> 1 - \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} \ldots \frac{1}{8}\right) + \ldots \left(\frac{1}{2^n - 1} - \frac{1}{2^n - 1} + \ldots \frac{1}{2^n - 1}\right)$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \ldots \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$= 1 + \frac{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \ldots \frac{1}{2}}{n\text{ times}} - \frac{1}{2^n}$$

$$= \left(1 - \frac{1}{2^n}\right) + \frac{n}{2}$$

Therefore, $a(200) > \left(1 - \frac{1}{2^{100}}\right) + \frac{200}{2} > 100$

Therefore, (D) is also the Ans.

29. $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3 (t-3)^5 \, dt$

$f'(x) = \frac{d}{dx} \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3 (t-3)^5 \, dt$

$= x(e^x - 1)(x-1)(x-2)^3 (x-3)^5 \times 1 - x(e^x - 1)(x-1)(x-2)^3 (x-3)^5 \times 0$

$$\frac{d}{dx} \int_{\phi(x)}^{\psi(x)} f(t)\, dt = f(\psi(x))\, \psi'(x) - f(\phi(x))\, \phi'(x) \text{ Formula}$$

For local minimum, $f'(x) = 0$

$\Rightarrow \quad x = 0, 1, 2, 3.$ Let $f'(x) = g(x) = x(e^x - 1)(x-1)(x-2)^3 (x-3)^5$

$g(x) =$ $-\infty$ (−) 0 (−) 1 (+) 2 (−) 3 (+) ∞

$\Rightarrow \quad f'(x) < 0 \quad$ if $\quad x < 0$

$< 0 \quad$ if $\quad 0 < x < 1$

$> 0 \quad$ if $\quad 1 < x < 2$

$< 0 \quad$ if $\quad 2 < x < 3$

$> 0 \quad$ if $\quad x > 3$

This shows that $f(x)$ has a local minimum at $x = 1$ and $x = 3$

Therefore, (B) and (D) are the Answer.

30. $4x^2 + 9y^2 = 1$ (given) ...(1)

Differentiating w.r.t. x, we get

$$8x + 18y \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}$$

The tangent at point (h, k) will be parallel to $8x = 9y$, then

$$\frac{-4h}{9k} = \frac{8}{9}$$

$\Rightarrow \quad h = -2k$ ...(2)

Substituting h, k in (1) since h, k lies in (1)

$$4h^2 + 9k^2 = 1$$

putting value of h in (2)

$$4(-2k)^2 - 9k^2 = 1$$

$$16k^2 + 9k^2 = 1 \quad \Rightarrow \quad 25k^2 = 1 \quad \Rightarrow \quad k^2 = 1/25$$

$$\Rightarrow \quad k = \pm 1/5$$

Thus, the points where the tangents are parallel to $8x = 9y$ are $(-2/5, 1/5)$ and $(2/5, -1/5)$. Therefore (B) (D) are the Ans.

31. Let A, B and C respectively denote the events that the student passes in Maths, Physics and Chemistry.

It is given :

$$P(A) = m, \quad P(B) = p \text{ and } P(C) = c$$

and P (passing in at least one's) = $P(A \cup B \cup C) = 0 \cdot 75$

$\Rightarrow 1 - P(A' \cap B' \cap C') = 0 \cdot 75$
$\Rightarrow 1 - P(A') \cdot P(B') \cdot P(C') = 0 \cdot 75$

$\because P(A) - 1 - P(\overline{A})$ and
$P(\overline{A \cup B \cup C}) = P(A \cap B \cap C)$

$\because$ A, B, C are independent events therefore, A', B' and C' are independent events.

$\Rightarrow \quad 0 \cdot 75 = 1 - (1 - m)(1 - p)(1 - c)$

$\Rightarrow \quad 0 \cdot 25 - (1 - m)(1 - p)(1 - c)$ ...(1)

Also P (passing exactly in two subjects) = $0 \cdot 4$

$\Rightarrow \quad P(A \cap B \cap \overline{C} \cup A \cap \overline{B} \cap C \cup \overline{A} \cap B \cap C) = 0 \cdot 4$

$\Rightarrow \quad P(A \cap B \cap \overline{C}) \cup P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) = 0 \cdot 4$

$\Rightarrow \quad P(A) \cdot P(B) \cdot P(\overline{C}) + P(A) P(\overline{B}) P(C) + P(\overline{A}) P(B) P(C) = 0 \cdot 4$

$\Rightarrow \quad p\, m(1 - c) + p(1 - m) c - (1 - p) mc = 0 \cdot 4$

$\Rightarrow \quad pm - pmc - pc - pmc + mc - pmc = 0 \cdot 4$ ...(2)

again P (passing at least in two subjects) $- 0 \cdot 5$

$\Rightarrow \quad P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C)$
$+ P(A \cap B \cap C) - 0 \cdot 5$

$\Rightarrow \quad pm(1 - c) + pc(1 - m) + cm(1 - p) + pcm = 0 \cdot 5$

$\Rightarrow \quad pm - pcm + pc - pcm + cm - pcm + pcm = 0 \cdot 5$

$\Rightarrow \quad (pm + pc + mc) - pcm = 0 \cdot 5$ ...(3)

from (2), we get

$pm + pc + mc - 3pcm = 0 \cdot 4$ ...(4)

from (1) we get

$0 \cdot 25 = 1 - (m + p + c) + (pm - pc + cm) - pmc$ ...(5)

solving (3), (4), (5) we get

$p + m + c = 1 \cdot 35 = 27/20$. Therefore, (B) is the Answer.

32. $y^2 = 2c(x + \sqrt{c})$ ...(1)

Differentiating w.r.t. x we get

$$2y \frac{dy}{dx} = 2c \qquad \Rightarrow \qquad c = y \frac{dy}{dx}$$

putting this value of c in (1) we get

$$y^2 = 2y \frac{dy}{dx}\left(x + \sqrt{y \frac{dy}{dx}}\right)$$

$$\Rightarrow \qquad y = 2 \frac{dy}{dx} \cdot x + 2y^{1/2} \left(\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \qquad y - 2x \frac{dy}{dx} = 2\sqrt{y} \left(\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \qquad \left(y - 2x \frac{dy}{dx}\right)^2 = 8y^3 \left(\frac{dy}{dx}\right)^3$$

Therefore, order of this differential equation is 1 and degree is 3.

Therefore (A), (C) is the Ans.

33. **Imp. note :** sequence is imp. for future considration in IIT exam.

Let a_n denote the length of side of the square S_n.

We are given a_n = length of diagonal of S_{n+1}.

$$\Rightarrow \quad a_n = \sqrt{2}\, a_{n+1}$$

$$\Rightarrow \quad a_{n-1} = \frac{a_n}{\sqrt{2}}$$

This shows that a_1, a_2, a_3 ....form a G.P. with common ratio $1/\sqrt{2}$.

Therefore, $a_n = a_1 \left(\frac{1}{\sqrt{2}}\right)^{n-1}$

$$\Rightarrow \quad a_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} \qquad \because a_1 = 10 \text{ given,}$$

$$\Rightarrow \quad a_n^2 = 100\left(\frac{1}{\sqrt{2}}\right)^{2(n-1)}$$

$$\Rightarrow \quad \frac{100}{2^{n-1}} \le 1 \qquad \because a_n^2 \le 1 \text{ given}$$

$$\Rightarrow \quad 100 \le 2^{n-1}$$

This is possible for $n \ge 8$.

so (B), (C), (D) are the Answer.

34. **Case 1.** $m = 0$

In this case $y = x - x^2$ ...(1)

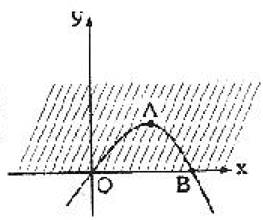
$y = 0$ ...(2)

are two given curves. $y > 0$ is total region above x-axis.

Therefore, area between

$y = x - x^2$ and $y = 0$

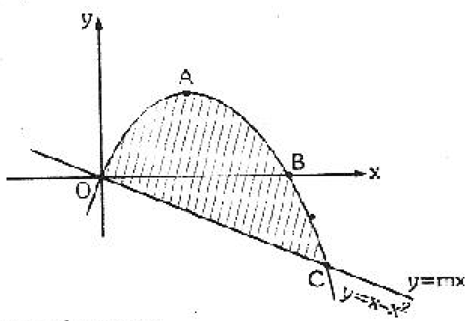
is area between $y = x - x^2$ and above the x-axis.



$$A = \int_0^1 (x - x^2)\, dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \ne \frac{9}{2}$$

Hence no solution.

Case 2. $m < 0$



In this case area between

$y = x - x^2$ and $y = mx$ is

$OABCO$ and points of intersection are (0, 0) and $[1 - m, \; m(1 - m)]$

Area $OABCO = \int_0^{1-m} [x - x^2 - mx]\, dx$.

Imp. note : Area $OBCO$ considered automatically because m is a parameter

$$= \left[(1 - m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m}$$

$$= \frac{1}{2}(1 - m)^3 - \frac{1}{3}(1 - m)^3$$

$$= \frac{1}{6}(1 - m)^3$$

Put Area $OABCO = 9/2$ (given)

$\therefore \quad \frac{1}{6}(1 - m)^3 = \frac{9}{2}$

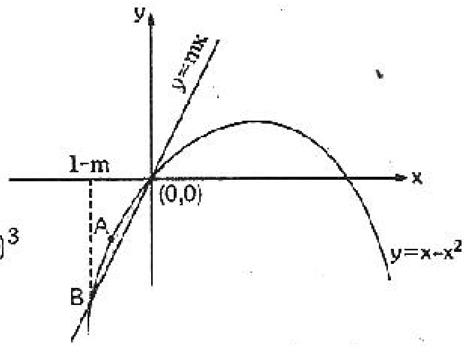
$\Rightarrow \quad (1 - m)^3 = 27$

$\Rightarrow \quad 1 - m = 3$

$\Rightarrow \quad m = -2$

Case 3. $m > 0$

In this case $y = mx$ and $y = x - x^2$ intersect in (0, 0) and $((1 - m), m(1 - m))$ as shown in Fig.



Area of shaded region

$$= \int_{1-m}^{0} (x - x^2 - mx)\, dx$$

$$= \left[(1 - m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0}$$

$$= -\frac{1}{2}(1 - m)(1 - m)^2 + \frac{1}{3}(1 - m)^3$$

$$= -\frac{1}{6}(1 - m)^3$$

Area of shaded region = 9/2 square unit (given)

$\Rightarrow \quad \frac{9}{2} = -\frac{1}{6}(1 - m)^3$

$\Rightarrow \quad (1 - m)^3 = -27$

$\Rightarrow \quad (1 - m) = -3$

$\Rightarrow \quad m = 3 + 1 = 4$. Therefore, (B) and (D) are the Answers.

35. **Imp. note : multiplicative loop** is very imp. approch in IIT mathematics.

$$\left(\tan\frac{\theta}{2}\right)(1 + \sec\theta) = \frac{\sin\theta/2}{\cos\theta/2}\cdot\left[1 + \frac{1}{\cos\theta}\right]$$

$$= \frac{\sin\theta/2}{\cos\theta/2} \times \frac{(1 + \cos\theta)}{\cos\theta} = \frac{\sin\theta/2 \cdot 2\cos^2\theta/2}{\cos\theta/2 \cdot \cos\theta}$$

$$= \frac{2\sin\theta/2 \cdot \cos\theta/2}{\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$